On the power of nonlocal boxes

or how nonlocality and entanglement are fundamentally different resources

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Nonlocality

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- The participants are individually given a challenge. In response, they must each produce an output.
- We say that the participant's outputs exhibit *nonlocality* if there is no classical theory that can explain the correlations of their outputs. Nonlocality can be achieved, for example, if the participants share entanglement.

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Two examples of nonlocal tasks

Pseudo-telepathy

- Consider a *game*, in which two or more players play as a team, against a *referee*.

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Pseudo-telepathy

- Consider a *game*, in which two or more players play as a team, against a *referee*.
- The players are physically separated and unable to communicate.
- They each receive an input ($x \in X$ for Alice, $y \in Y$ for Bob).
- They must each produce output ($a \in A$ for Alice, $b \in B$ for Bob) such that a given *winning condition* (a relation R on $X \times Y \times A \times B$) is satisfied. If R is satisfied, we say that the players *win* the game.

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Factoid

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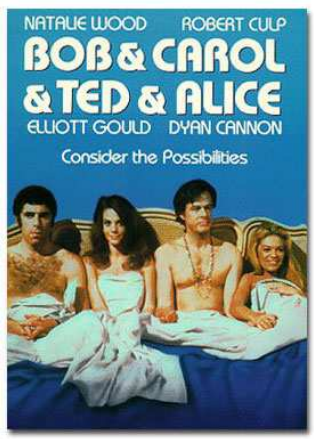
- According to the Wikipedia, “Alice” and “Bob” were invented by Ron Rivest for the 1978 Communications of the ACM article presenting the RSA cryptosystem.
- Rivest denies that these names have any relation with the 1969 movie “Bob & Carol & Ted & Alice” as some suggest.



Pseudo-telepathy

- We say that the players have a *winning strategy* if they can win on all possible inputs. A winning strategy can be *classical* (players share only classical resources), *quantum* (players share entanglement), or *nonlocal* (players share nonlocal boxes, more on this later).

Pseudo-telepathy

- We say that the players have a *winning strategy* if they can win on all possible inputs. A winning strategy can be *classical* (players share only classical resources), *quantum* (players share entanglement), or *nonlocal* (players share nonlocal boxes, more on this later).
- A game exhibits *pseudo-telepathy* if it admits a quantum winning strategy and does not admit a classical winning strategy.

Pseudo-telepathy

- *Theorem*: No pseudo-telepathy game exists where the quantum strategy makes use of a single pair of entangled qubits. (Brassard, Méthot, Tapp, 2005)

Entanglement simulation

- Entanglement simulation is the exact reproduction of the correlations of quantum entanglement by participants who do not have access to quantum entanglement. An additional resource, such as communication, is usually required.

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Simulation and pseudo-telepathy

- A protocol *simulates* the correlations of a pseudo-telepathy game if, in addition to yielding a winning strategy, the outputs are indistinguishable from the outputs of the quantum winning strategy.

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Simulation and pseudo-telepathy

- A protocol *simulates* the correlations of a pseudo-telepathy game if, in addition to yielding a winning strategy, the outputs are indistinguishable from the outputs of the quantum winning strategy.
- Simulating the entangled state used in the quantum winning strategy cannot be any easier than simulating the correlations of a given pseudo-telepathy game.
- This gives us a lower bound on the amount of resources required for entanglement simulation.

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The nonlocal box

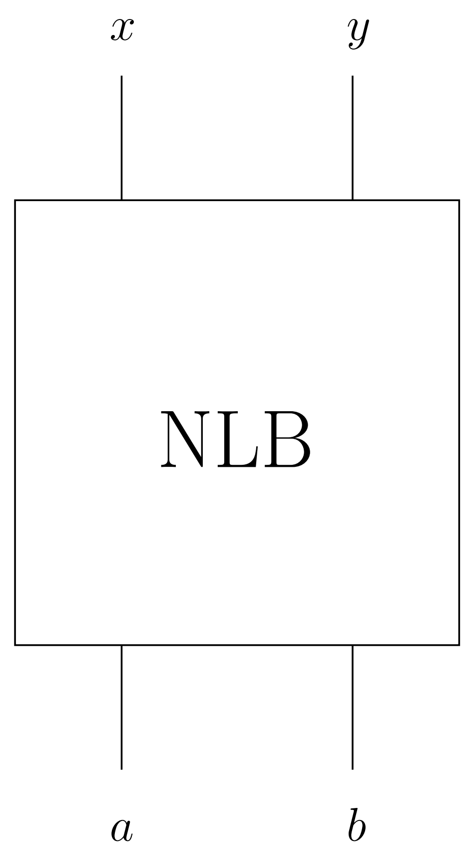
The nonlocal box (NLB)

A virtual device shared between two participants, Alice and Bob. When Alice inputs a bit x and Bob inputs a bit y, Alice receives a bit a and Bob a bit b such that:

$$a \oplus b = x \wedge y.$$

- Furthermore, a and b are uniformly distributed among all solutions.

The nonlocal box (NLB)



Properties of the Nonlocal Box

- Cannot be used for signaling

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Properties of the Nonlocal Box

- Cannot be used for signaling
  - a and b are uniformly distributed
- Cannot be reproduced by classical participants
  - Alice and Bob can't communicate
- Cannot be reproduced by quantum participants
  - Result due to Tsirelson(1980)

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Properties of the Nonlocal Box

- The NLB was defined by Popescu and Rohrlich (1994).

Properties of the Nonlocal Box

- The NLB was defined by Popescu and Rohrlich (1994).
- A single use of a NLB is sufficient for the simulation of a maximally-entangled 2-qubit state, for example, $|\psi^-\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$ (Cerf, Gisin, Massar, Popescu 2004).

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Measures of nonlocality

- How can we measure the *nonlocalness* of a specific task?

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  - Minimum detector efficiency rate required in order to circumvent the detection loophole.
  - Number of bits of communication required in order for classical players to succeed.
  - *Number of NLB uses required in order for classical players to succeed.*

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Non-local Winning Strategies for Pseudo-Telepathy Games

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The Magic Square Game

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Magic Square

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- A simple parity argument shows that no magic square exists.

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The Game

- Alice's input is *row* $x \in \{1, 2, 3\}$.
  Bob's input is *column* $y \in \{1, 2, 3\}$.

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- Alice outputs a, corresponding to row x of a magic square.
  Bob outputs b, corresponding to column y of a magic square.

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  Bob's input is *column* $y \in \{1,2,3\}$.
- Alice outputs a, corresponding to row x of a magic square.
  Bob outputs b, corresponding to column y of a magic square.
- The intersection of Alice and Bob's answers must agree.

		0
		1
1	1	1

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Properties

- No classical winning strategy

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  - such a strategy corresponds to a magic square

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Properties

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- no quantum winning strategy using a 2-qubit entangled state

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Properties

- No classical winning strategy
  - such a strategy corresponds to a magic square
- no quantum winning strategy using a 2-qubit entangled state
  - no such pseudo-telepathy game exists
- there exists a quantum winning strategy using a 4-qubit entangled state

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Properties

- No classical winning strategy
  - such a strategy corresponds to a magic square
- no quantum winning strategy using a 2-qubit entangled state
  - no such pseudo-telepathy game exists
- there exists a quantum winning strategy using a 4-qubit entangled state
  - quantum strategy (Aravind, 2002) using $\frac{1}{2}|0011\rangle - \frac{1}{2}|0110\rangle - \frac{1}{2}|1001\rangle + \frac{1}{2}|1100\rangle$

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Nonlocal players

- There exists a nonlocal winning strategy for the magic square game that makes use of a single NLB.

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Nonlocal players

- There exists a nonlocal winning strategy for the magic square game that makes use of a single NLB.
- Proof: Alice and Bob each have two strategies, $A0$ and $A1$ for Alice and $B0$ and $B1$ for Bob such that:

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Nonlocal players

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- Both pairs of strategies $(A0, B1)$ and $(A1, B0)$ yield a correct answer when $x = y = 3$.
- Now, Alice and Bob each input 1 into the NLB if their input is 3 (and otherwise they input 0). They use the output of the NLB to determine which strategy to use.

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Nonlocal players

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Nonlocal players

- In the quantum winning strategy, the outcomes of the players are uniformly distributed.
- By randomizing over all possible strategies $A0, A1, B0, B1$, it is possible to simulate the correlations of the Magic Square game.
- Corollary: A NLB can simulate bipartite correlations that no 2-qubit entangled state, $\alpha|00\rangle + \beta|11\rangle$, can.

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The Mermin-GHZ Game

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The Game

- In the Mermin-GHZ pseudo-telepathy game:

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The Game

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- Alice, Bob and Charlie receive as input a single bit, x, y and z, respectively.

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- There is a *promise* that $x \oplus y \oplus z = 0$.

The Game

- In the Mermin-GHZ pseudo-telepathy game:
- Alice, Bob and Charlie receive as input a single bit, x, y and z, respectively.
- There is a *promise* that $x \oplus y \oplus z = 0$.
- Alice and Bob must output one bit each, a, b and c respectively, such that $a \oplus b \oplus c = \frac{x+y+z}{2}$.

The Game

- In a classical strategy, suppose Alice, Bob and Charlie output a_i, b_i and c_i respectively, on input i. This is a winning strategy if and only if the following system of equations is satisfied:

$$a_0 \oplus b_0 \oplus c_0 = 0$$
$$a_0 \oplus b_1 \oplus c_1 = 1$$
$$a_1 \oplus b_0 \oplus c_1 = 1$$
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The Game

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- Again, a simple parity argument shows that no such strategy exists.

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Properties

- no classical winning strategy

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Properties

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Properties

- no classical winning strategy
  - simple parity argument
- no quantum winning strategy using a 2-qubit entangled state
  - no such pseudo-telepathy game exists
- there exists a quantum winning strategy using an 3-qubit entangled state

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Properties

- no classical winning strategy
  - simple parity argument
- no quantum winning strategy using a 2-qubit entangled state
  - no such pseudo-telepathy game exists
- there exists a quantum winning strategy using an 3-qubit entangled state
  - quantum strategy using $\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$ (Greenberger, Horne, Zeilinger, 1989)

Theorem

- Theorem: There exists a nonlocal winning strategy for the Mermin-GHZ game that makes use of a single NLB.

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Theorem

- Theorem: There exists a nonlocal winning strategy for the Mermin-GHZ game that makes use of a single NLB.
- Proof: Alice and Bob input $\overline{x}$ and $\overline{y}$ into a NLB. They set a and b as their respective outcomes of the NLB. Charlie simply outputs $c = 1$. Taking into account the promise, it is easy to see that this strategy works.

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Theorem

x	y	z	$\bar{x}$	$\bar{y}$	$a \oplus b$	c	$a \oplus b \oplus c$	$\frac{x+y+z}{2}$
0	0	0	1	1	1	1	0	0
0	1	1	1	0	0	1	1	1
1	0	1	0	1	0	1	1	1
1	1	0	0	0	0	1	1	1

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Corollary

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- If Bob and Charlie share a random bit r and they output $b \oplus r$ and $b \oplus r$ respectively, then we have a simulation of the correlations of the Mermin-GHZ game.

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Corollary

- In the quantum winning strategy, the outcomes of the players are uniformly distributed.
- If Bob and Charlie share a random bit r and they output $b \oplus r$ and $b \oplus r$ respectively, then we have a simulation of the correlations of the Mermin-GHZ game.
- Corollary: A NLB can simulate tripartite correlations that no 2-qubit entangled state, $\alpha|00\rangle + \beta|11\rangle$, can.

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Nonlocality and Entanglement are different

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Previous and new results

- Theorem: there exist bipartite entangled states of two qubits that *cannot* be simulated with a single use of a NLB. (Brunner, Gisin, Scarani 2005)

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- “entanglement and nonlocality are different resources”...or are they?

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- “entanglement and nonlocality are different resources”...or are they?
- The result is not *asymptotic*. It does not rule out the possibility that $O(n)$ NLB are sufficient to simulate n bipartite 2-qubit entangled states.

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- "entanglement and nonlocality are different resources"...or are they?
- The result is not *asymptotic*. It does not rule out the possibility that $O(n)$ NLB are sufficient to simulate n bipartite 2-qubit entangled states.
- Here, we show that entanglement and nonlocality are *asymptotically* different.

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Distributed Deutsch-Jozsa game

- Alice and Bob each receive a 2^n-bit string, x and y.

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Distributed Deutsch-Jozsa game

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- There is a promise that $\Delta(x,y) \in \{0, 2^{n-1}\}$.
- Alice and Bob must each output an n-bit string a and b such that $[a = b] \Leftrightarrow [x = y]$.

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Properties of the game

- For all $n \geq 4$, this is a pseudo-telepathy game (Newman, 2004).

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Properties of the game

- For all $n \geq 4$, this is a pseudo-telepathy game (Newman, 2004).
- The quantum state used for the quantum winning strategy is $\frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} |j\rangle|j\rangle$.
- A classical winning strategy for the game requires $\Omega(2^n)$ bits of communication (Brassard, Cleve, Tapp, 1999).

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Theorem

- Theorem: For the distributed Deutsch-Jozsa pseudo-telepathy game, $\Omega(2^n)$ NLB uses are required in a nonlocal winning strategy.

Theorem

- Theorem: For the distributed Deutsch-Jozsa pseudo-telepathy game, $\Omega(2^n)$ NLB uses are required in a nonlocal winning strategy.
- Proof: If we had a nonlocal winning strategy with less than $\Omega(2^n)$ NLB uses, we could use communication to get a classical winning strategy with less than $\Omega(2^n)$ bits of communication, which is a contradiction.

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Asymptotic result!

- We have shown: there exists a state of n maximally entangled bipartite states of two qubits that requires at least 2^n NLB uses to simulate.

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Asymptotic result!

- We have shown: there exists a state of n maximally entangled bipartite states of two qubits that requires at least 2^n NLB uses to simulate.
- Entanglement and nonlocality are fundamentally different resources after all!

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NLB pseudo-telepathy

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- Recall that a pseudo-telepathy game is one which does not admit a classical winning strategy, whereas a quantum winning strategy does exist.

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- A *NLB pseudo-telepathy* game is one which does not admit a quantum winning strategy, whereas a nonlocal winning strategy exists.

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NLB pseudo-telepathy

- Recall that a pseudo-telepathy game is one which does not admit a classical winning strategy, whereas a quantum winning strategy does exist.
- A *NLB pseudo-telepathy* game is one which does not admit a quantum winning strategy, whereas a nonlocal winning strategy exists.
- We have already seen an example of a NLB pseudo-telepathy game: the NLB itself!

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A new multi-party NLB

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- Each of the n participants receives an input bit.
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A new multi-party NLB

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- Each of the n participants receives an input bit.
- They must each produce an output bit such that the $\oplus$ of all the outputs is equal to the *majority* of the input bits.
- This new multi-party NLB is a generalization of a two-party NLB.

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Properties of the new NLB

- no classical winning strategy

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Properties of the new NLB

- no classical winning strategy
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Properties of the new NLB

- no classical winning strategy
  - the NLB is a special case of this game
- no quantum winning strategy
  - the NLB is a special case of this game
- $\Omega(n)$ NLBs are necessary in a nonlocal winning strategy
  - each player must be linked to another through a NLB

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Conclusion and Future Work

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Recap

We have made progress towards characterizing the power of the NLB.

- A single NLB can simulate correlations that no entangled pair of qubits can; in the bipartite and in the tri-partite scenario.

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- nonlocality and entanglement are fundamentally different resources: there exists correlation whose simulation requires an exponential amount of NLB uses.

Recap

We have made progress towards characterizing the power of the NLB.

- A single NLB can simulate correlations that no entangled pair of qubits can; in the bipartite and in the tri-partite scenario.
- nonlocality and entanglement are fundamentally different resources: there exists correlation whose simulation requires an exponential amount of NLB uses.
- We have defined non-local pseudo-telepathy and proposed a multi-party NLB.

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Future Work

- Finding nonlocal winning strategies for all pseudo-telepathy games, or showing that such a task is impossible.

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Future Work

- Finding nonlocal winning strategies for all pseudo-telepathy games, or showing that such a task is impossible.
- Finding applications for the new multi-party nonlocal box (for instance, in multi-party entanglement simulation).

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