PIMS-MITACS
Summer School on Quantum Information Science

Non locality

June 2003

Alain Tapp

IRO Université de Montréal

Speed of communication

« Faster than light communication is absolutely impossible»

« The independent realisation of an experiment in a far away location should not instantaneously influence the outcome of an experiment in my lab»

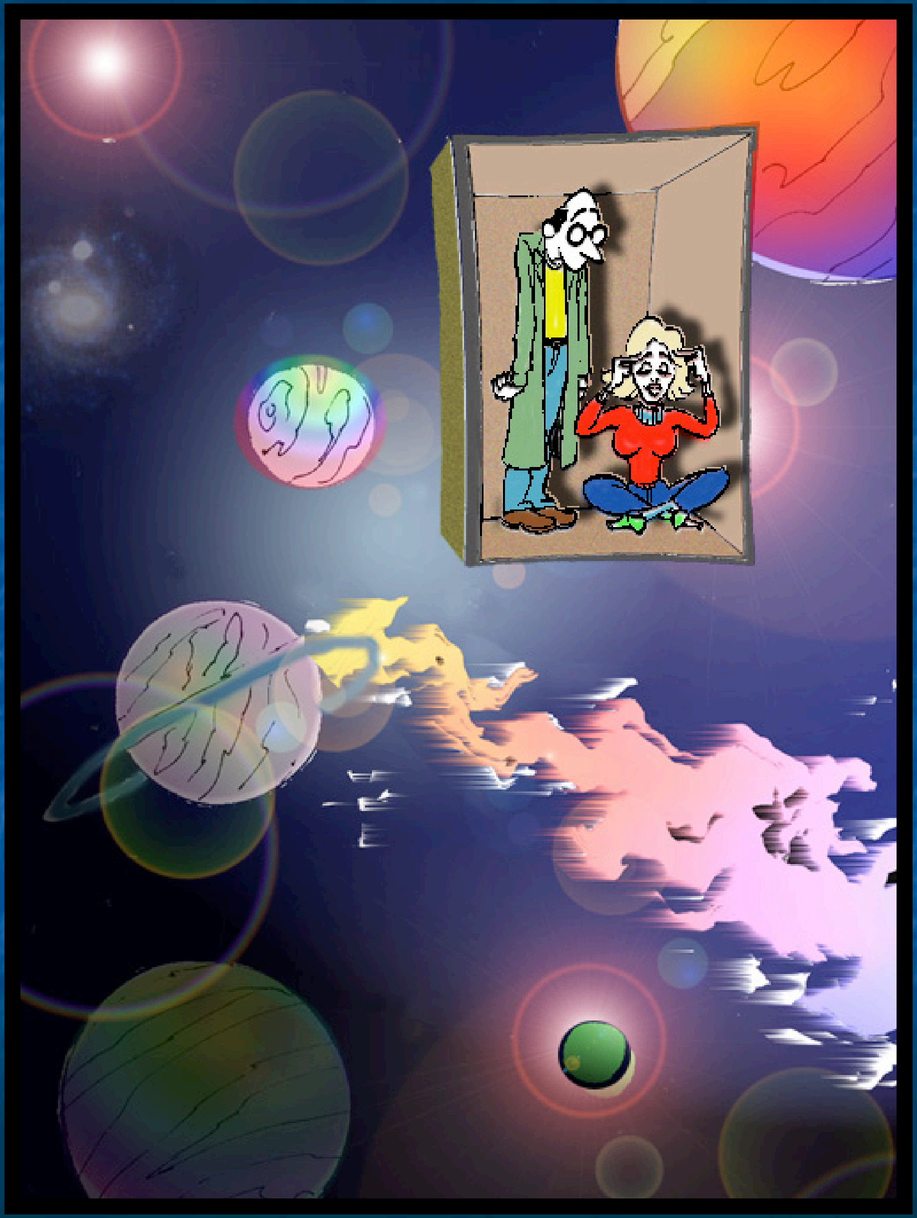
«The outcome of a measurement carried out on a system only depend on that system»

Pseudo-telepathy

- « The capacity to perform a task that requires communication without signaling »

Setup stage

The game



The game

	Alice	Bob
Questions (*1* bit)	$x_A \in \{A, B\}$	$x_B \in \{B, C\}$
Answer (*1* bit)	$y_A \in \{0,1\}$	$y_B \in \{0,1\}$

Rule of the game: If $x_A = x_B = B$ then $y_A = y_B$

The game is to be repeated numerous times.

$$P_{XY} = \Pr[y_A = y_B \mid x_A = X \text{ and } x_B = Y]$$

The goal is to maximize $P_{AB} + P_{BC} - P_{AC}$

The strategy

Assume Alice and Bob have agreed on *any* strategy that does not involve communication. Randomness is allowed.

In round i if $x_A=A$ Alice outputs a_i.
In round i if $x_A=B$ Alice outputs b_i.
In round i if $x_B=B$ Bob outputs b_i.
In round i if $x_B=C$ Bob outputs c_i.

The inequality

$$\Pr[a=b=c]+\Pr[a=b\neq c]+\Pr[a\neq b=c]+\Pr[a\neq b\neq c]=1$$

$$\begin{aligned}
&\Pr[a=b]+\Pr[b=c]-\Pr[a=c]\\
&=\left(\Pr[a=b=c]+\Pr[a=b\neq c]\right)+\left(\Pr[a=b=c]+\Pr[a\neq b=c]\right)-\left(\Pr[a=b=c]+\Pr[a\neq b\neq c]\right)\\
&=\Pr[a=b=c]+\Pr[a=b\neq c]+\Pr[a\neq b=c]-\Pr[a\neq b\neq c]\\
&\leq\Pr[a=b=c]+\Pr[a=b\neq c]+\Pr[a\neq b=c]+\Pr[a\neq b\neq c]\\
&=1
\end{aligned}$$

$$\Pr[a=b]+\Pr[b=c]-\Pr[a=c]\leq 1$$

$$P_{AB}+P_{BC}-P_{AC}\leq 1$$

The contradiction

Alice and Bob are separated (several lightyears away) and numerous instances of the test are performed. The collected data shows the following correlations:

$$P_{AB} = P_{BC} = \frac{3}{4} \qquad P_{AC} = \frac{1}{4}$$

$$P_{AB} + P_{BC} - P_{AC} = \frac{5}{4} > 1$$

Pseudo-telepathy is then possible!
Is something wrong?

Quantum strategy

For each round Alice and Bob share a precious pair of entangled qubits.

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$$

$$U|0\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle \qquad U^{-1}|0\rangle = -\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

$$U|1\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle \qquad U^{-1}|1\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

If $x_A=A$ Alice performs U.
If $x_B=C$ Bob performs U^{-1}.
Alice and Bob measure the state and output the resulting bit.

Quantum strategy

If $x_A = B$ and $x_B = B$ then no unitary transformation is performed and the players measure

$$|\square\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$$

and obtain the same result with probability 1.

Quantum strategy

If $x_A = A$ and $x_B = B$ then Alice performs U on her qubit.

$$U \otimes I|\Psi\rangle = \frac{1}{\sqrt{2}}\left(U \otimes I|00\rangle_{AB} + U \otimes I|11\rangle_{AB}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\left(\frac{\sqrt{3}}{2}|0\rangle_A + \frac{1}{2}|1\rangle_A\right)|0\rangle_B + \left(\frac{1}{2}|0\rangle_A - \frac{\sqrt{3}}{2}|1\rangle_A\right)|1\rangle_B\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{2}}|00\rangle_{AB} + \frac{1}{2\sqrt{2}}|10\rangle_{AB} + \frac{1}{2\sqrt{2}}|01\rangle_{AB} - \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle_{AB}$$

$$\Pr[y_A = y_B] = \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^2 + \left(-\frac{\sqrt{3}}{2\sqrt{2}}\right)^2 = \frac{3}{4}$$

Quantum strategy

If $x_A = B$ and $x_B = C$ then Bob perform U^{-1} on his qubit.

$$I \otimes U^{-1}|\psi\rangle = \frac{1}{\sqrt{2}}\left(I \otimes U^{-1}|00\rangle_{AB} + I \otimes U^{-1}|11\rangle_{AB}\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle_A\left(-\frac{\sqrt{3}}{2}|0\rangle_B + \frac{1}{2}|1\rangle_B\right) + |1\rangle_B\left(\frac{1}{2}|0\rangle_A + \frac{\sqrt{3}}{2}|1\rangle_A\right)\right)$$

$$= -\frac{\sqrt{3}}{2\sqrt{2}}|00\rangle_{AB} + \frac{1}{2\sqrt{2}}|01\rangle_{AB} + \frac{1}{2\sqrt{2}}|10\rangle_{AB} + \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle_{AB}$$

$$\Pr[y_A = y_B] = \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^2 = \frac{3}{4}$$

Quantum strategy

If $x_A=A$ and $x_B=C$ then Alice performs U on her qubit and Bob performs U^{-1} on his.

$$U \otimes U^{-1}|\psi\rangle = \frac{1}{\sqrt{2}}\left(U \otimes U^{-1}|00\rangle_{AB} + U \otimes U^{-1}|11\rangle_{AB}\right)$$

$$= \frac{1}{\sqrt{2}}\left[\left(\frac{\sqrt{3}}{2}|0\rangle_A + \frac{1}{2}|1\rangle_A\right)\left(-\frac{\sqrt{3}}{2}|0\rangle_B + \frac{1}{2}|1\rangle_B\right) + \left(\frac{1}{2}|0\rangle_B - \frac{\sqrt{3}}{2}|1\rangle_B\right)\left(\frac{1}{2}|0\rangle_A + \frac{\sqrt{3}}{2}|1\rangle_A\right)\right]$$

$$= -\frac{1}{2\sqrt{2}}|00\rangle_{AB} + \frac{\sqrt{3}}{2\sqrt{2}}|01\rangle_{AB} - \frac{\sqrt{3}}{4\sqrt{2}}|10\rangle_{AB} - \frac{1}{2\sqrt{2}}|11\rangle_{AB}$$

$$\Pr[y_A = y_B] = \left(-\frac{1}{2\sqrt{2}}\right)^2 + \left(-\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{4}$$

Interpretation

With very high probability Alice and Bob will violate the inequality that we have derived. This will enable them to win the game and achieve the pseudo-telepathy trick.

Alice and Bob are not strictly speaking exchanging information. Neither one learns the other player's input.

This proves that God plays dice and that quantum mechanics is not a local theory.

Mermin's game

	Alice	Bob	Charlie
Questions (*1* bits)	x_A	x_B	x_C
Answers (*1* bits)	y_A	y_B	y_C

Promise: $x_A \oplus x_B \oplus x_C = 1$

Winning condition: $y_A \oplus y_B \oplus y_C = x_A \lor x_B \lor x_C$

$$\left.\begin{array}{l}(x_A = 0, x_B = 0, x_C = 1)\\ (x_A = 0, x_B = 1, x_C = 0)\\ (x_A = 1, x_B = 0, x_C = 0)\end{array}\right\}$$ Even

$(x_A = 1, x_B = 1, x_C = 1)$ Odd

Classical strategy?

Each player must have a strategy.

If $x_A = 0$ Alice outputs $y_A = a_0$
If $x_A = 1$ Alice outputs $y_A = a_1$

If $x_B = 0$ Bob outputs $y_B = b_0$
If $x_B = 1$ Bob outputs $y_B = b_1$

If $x_C = 0$ Charlie outputs $y_C = c_0$
If $x_C = 1$ Charlie outputs $y_C = c_1$

Alice, Bob and Charlie’s strategies must be consistent with the game’s constraints.

No classical strategy!

$$x_A = 0, x_B = 0 \text{ and } x_C = 1 \quad \Rightarrow \quad \begin{cases} 0 = x_A \oplus x_B \oplus x_C \\ 0 = y_A \oplus y_B \oplus y_C \\ 0 = a_0 \oplus b_0 \oplus c_1 \end{cases}$$

$$x_A = 0, x_B = 1 \text{ and } x_C = 0 \quad \Rightarrow \quad 0 = a_0 \oplus b_1 \oplus c_0$$

$$x_A = 1, x_B = 0 \text{ and } x_C = 0 \quad \Rightarrow \quad 0 = a_1 \oplus b_0 \oplus c_0$$

$$x_A = 1, x_B = 1 \text{ and } x_C = 1 \quad \Rightarrow \quad 1 = a_1 \oplus b_1 \oplus c_1$$

$$0 \oplus 0 \oplus 0 \oplus 1 = (a_0 \oplus b_0 \oplus c_1) \oplus (a_0 \oplus b_1 \oplus c_0) \oplus (a_1 \oplus b_0 \oplus c_0) \oplus (a_1 \oplus b_1 \oplus c_1)$$

$$1 = (a_0 \oplus a_0) \oplus (a_1 \oplus a_1) \oplus (b_0 \oplus b_0) \oplus (b_1 \oplus b_1) \oplus (c_0 \oplus c_0) \oplus (c_1 \oplus c_1)$$

$$1 = 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0$$

$$1 = 0$$

Classical players

Any classical strategy will be inconsistent with one of the four equations and therefor if the questions are chosen uniformly at random 25% of the time the players will fail the test.

Now suppose the players are separated by many lightyears but still succeed one million times in a row…

What should we conclude?
Pseudo-telepathy???

Quantum players

$$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \qquad H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$H\left(\alpha|0\rangle + \beta|1\rangle\right) = \alpha H|0\rangle + \beta H|1\rangle$$

$$= \alpha\frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$= \frac{\alpha + \beta}{\sqrt{2}}|0\rangle + \frac{\alpha - \beta}{\sqrt{2}}|1\rangle$$

$$HH\left(\alpha|0\rangle + \beta|1\rangle\right) = H\left(\frac{\alpha + \beta}{\sqrt{2}}|0\rangle + \frac{\alpha - \beta}{\sqrt{2}}|1\rangle\right) = \alpha|0\rangle + \beta|1\rangle$$

$$HH = I \qquad H = H^{-1}$$

$$|\psi\rangle = \frac{1}{\sqrt{8}}\left(|000\rangle + |001\rangle + |010\rangle + |100\rangle - |011\rangle - |101\rangle - |110\rangle - |111\rangle\right)$$

$$(H \quad I \quad I)\,|\psi\rangle = \frac{1}{2}\left(|000\rangle - |011\rangle + |101\rangle + |110\rangle\right)$$

$$(I \quad H \quad I)\,|\psi\rangle = \frac{1}{2}\left(|000\rangle + |011\rangle - |101\rangle + |110\rangle\right)$$

$$(I \quad I \quad H)\,|\psi\rangle = \frac{1}{2}\left(|000\rangle + |011\rangle + |101\rangle - |110\rangle\right)$$

Even

$$(H \quad H \quad H)\,|\psi\rangle = \frac{1}{2}\left(|001\rangle + |010\rangle + |100\rangle - |111\rangle\right)$$

Odd

The three players share the state $|\psi\rangle$
Each player applies H if his input is 1.
Each player measures his qubit and outputs the resulting bit.

$$(H \otimes I \otimes I)|\psi\rangle$$

$$= \frac{1}{\sqrt{8}}\left[\begin{array}{l}(H \otimes I \otimes I)|000\rangle + (H \otimes I \otimes I)|001\rangle + (H \otimes I \otimes I)|010\rangle + (H \otimes I \otimes I)|100\rangle \\ -(H \otimes I \otimes I)|011\rangle - (H \otimes I \otimes I)|101\rangle - (H \otimes I \otimes I)|110\rangle - (H \otimes I \otimes I)|111\rangle\end{array}\right]$$

$$= \frac{1}{\sqrt{8}}\left[\begin{array}{l}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|00\rangle + \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|01\rangle + \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|10\rangle + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|00\rangle \\ -\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|11\rangle - \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|01\rangle - \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|10\rangle - \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|11\rangle\end{array}\right]$$

$$= \frac{1}{4}\left[\begin{array}{l}|000\rangle + |100\rangle + |001\rangle + |101\rangle + |010\rangle + |110\rangle + |000\rangle - |100\rangle \\ -|011\rangle - |111\rangle - |001\rangle + |101\rangle - |010\rangle + |110\rangle - |011\rangle + |111\rangle\end{array}\right]$$

$$= \frac{1}{4}\left(2|000\rangle + 2|101\rangle + 2|110\rangle - 2|011\rangle\right)$$

$$= \frac{1}{2}\left(|000\rangle + |101\rangle + |110\rangle - |011\rangle\right)$$

Asymptotic result

There exists a pseudo-telepathy game that the players can win with probability 1 if they share *n* Bell states.

Without entanglement this game requires $\Omega(2^n)$ bits of communication.

This gives an exponential bound on the amount of communication required to simulate *n* entangled qubits.

[BCT99]