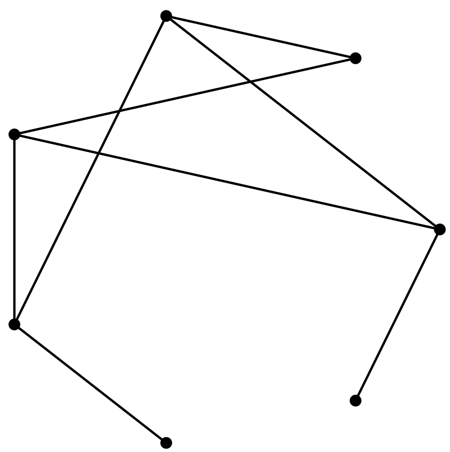
Spatial Search by Phased Continuous-time Quantum Walk

Heath Gerhardt

University of Calgary

Continuous-time Quantum Walks



$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- Graphs are described by adjacency matrices:
  $$A[u,v] = \begin{cases} 1 & \text{u,v connected} \\ 0 & \text{otherwise} \end{cases}$$
- Quantum systems can be described by a Hamiltonian H. $H[u,v]$ is non-zero if amplitude move from state i to state j.
- We want to create a quantum system whose dynamics follows the structures of the graph.

Continuous-time Quantum Walks

$$H = A$$

[Farhi, Gutmann 1998]

- Dynamics follows the Schrodinger equation

$$H|\psi\rangle = i\frac{d}{dt}|\psi\rangle$$

- Or as a unitary operator

$$U(t_1, t_2) = e^{-iH(t_2 - t_1)}$$

Phased Cont.-time Quantum Walks

- In general Hamiltonians can be arbitrary Hermitian matrices,

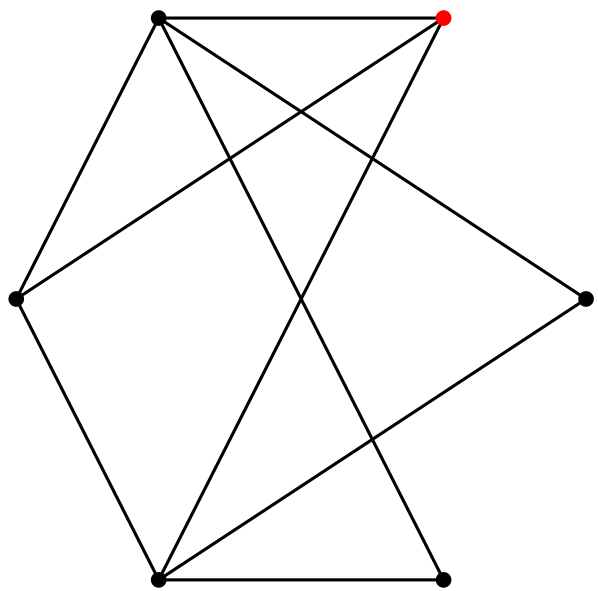
$$H = H^\dagger$$

  and so can have complex entries.
- If $A[u, v] = 1$ why restrict ourselves to $H[u, v] = 1$ when we are dealing with quantum systems?
- Why not allow $H[u, v] = e^{i\theta}$? (the most general would be $H[u, v] = re^{i\theta}$ but we are not dealing with weighted graphs)

Phased Cont.-time Quantum Walks

- **Phased Cont-time Quantum Walks:**
  Let A be the adjacency matrix of a graph G with n vertices, the phased continuous-time quantum walk on G is defined by
  $$H[u,v] = e^{i\theta_j}$$
  for $j \in \{1, \ldots, n\}$, and every edge (u, v) of G.
- What is the justification for introducing this model?
- Lets think about a quantum network...

Justifying Phased Walks



$M = e^{r(D-A)t}$

Classically we jump from one vertex to another after some random amount of time.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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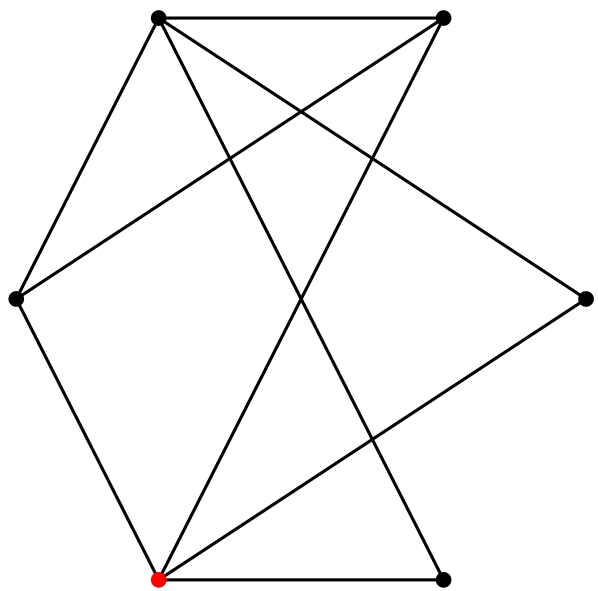

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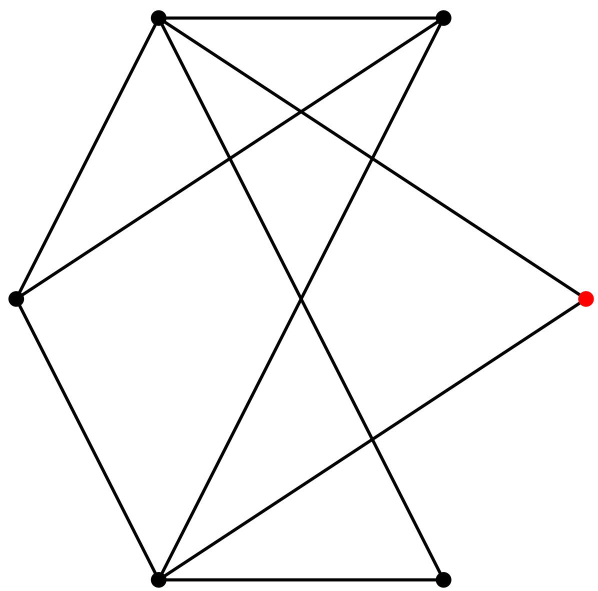

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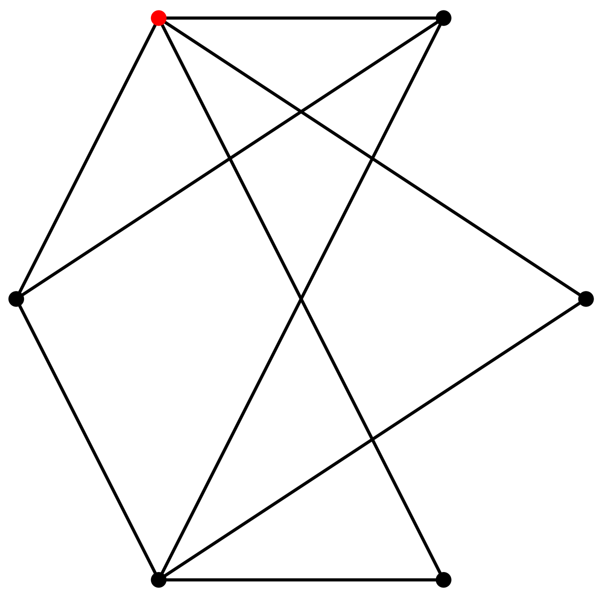

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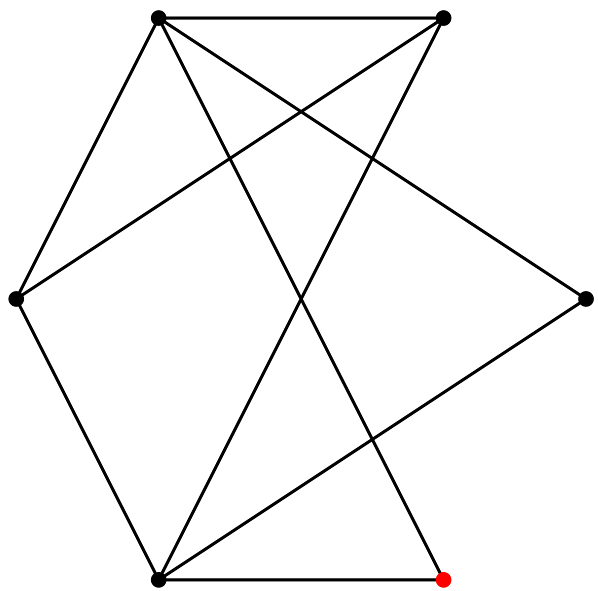

$$
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
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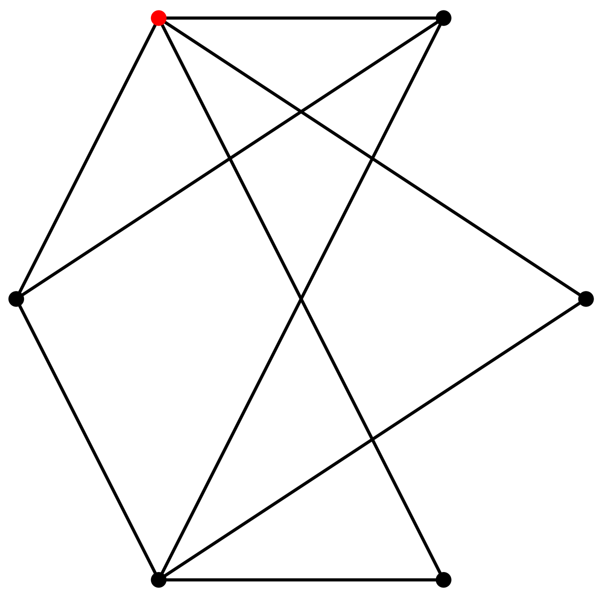

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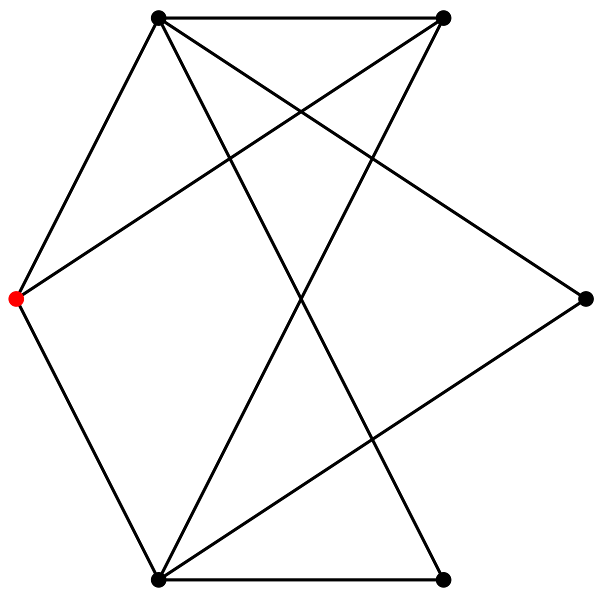
Justifying Phased Walks



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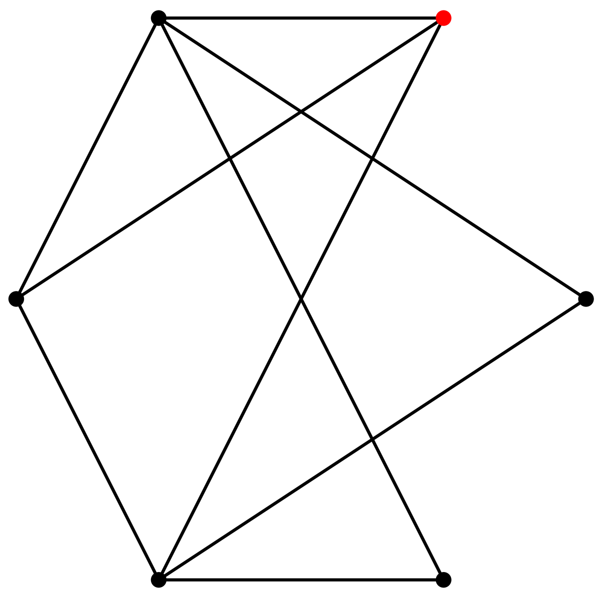

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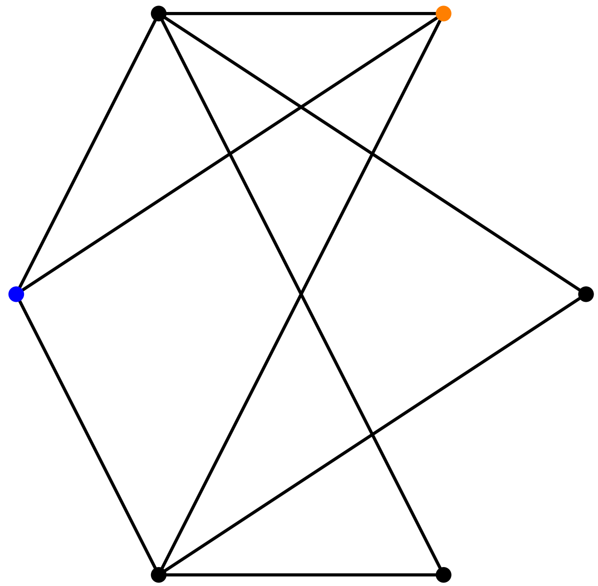
$U = e^{i(D-A)t}$

Quantumly amplitude diffuses through the graph.

$$t = 0.0$$

{1,0,0,0,0,0}

Justifying Phased Walks



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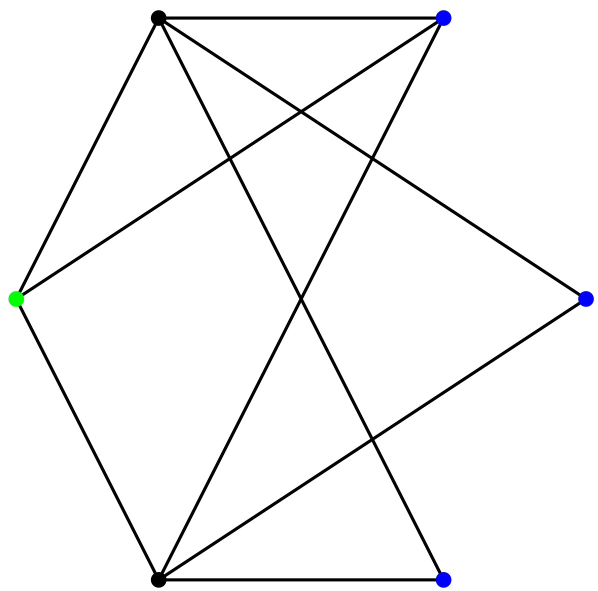
$U = e^{i(D-A)t}$

Quantumly amplitude diffuses through the graph.

$$t = 0.5$$

{0.468, 0.119, 0.205, 0.119, 0.0438, 0.043}

Justifying Phased Walks



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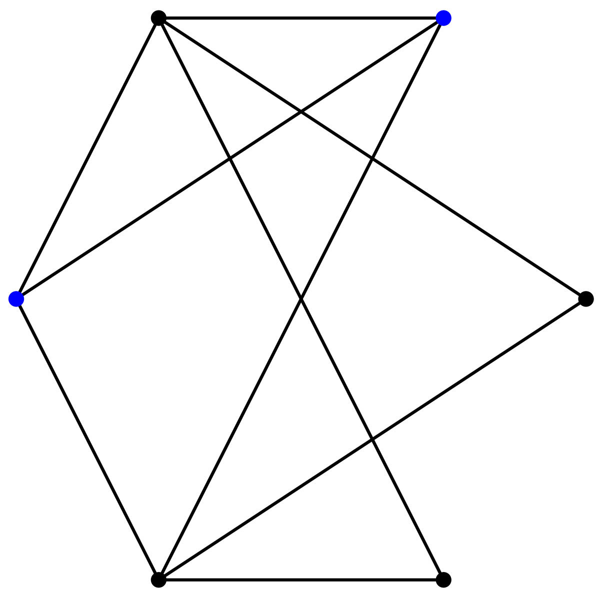
$U = e^{i(D-A)t}$

Quantumly amplitude diffuses through the graph.

$$t = 1.0$$

{0.160, 0.018, 0.346, 0.018, 0.223, 0.223}

Justifying Phased Walks



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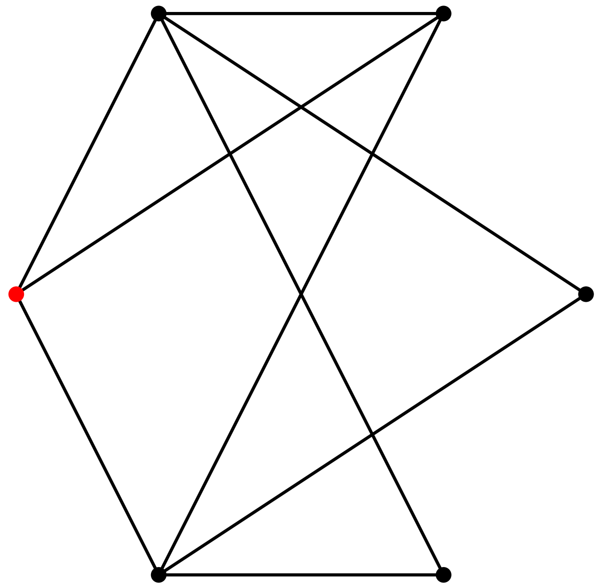
$U = e^{i(D-A)t}$

Quantumly amplitude diffuses through the graph.

$$t = 1.5$$

{0.253, 0.125, 0.272, 0.125, 0.111, 0.111}

Justifying Phased Walks



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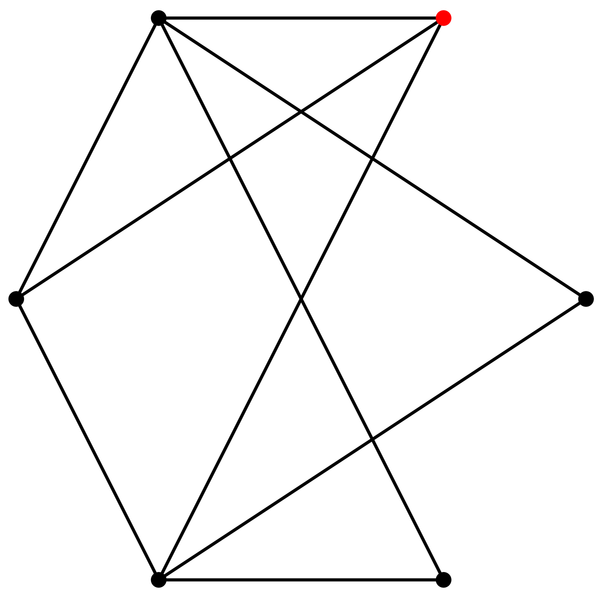
$U = e^{i(D-A)t}$

Quantumly amplitude diffuses through the graph.

$$t = 2.0$$

{0.062, 0.039, 0.831, 0.039, 0.013, 0.013}

Justifying Phased Walks



$U = e^{-iPt}$

Random phases give different dynamics.

$$t = 0.0$$

{1,0,0,0,0,0}

$$\begin{pmatrix} 0 & e^{-i\theta_1} & e^{-i\theta_2} & e^{-i\theta_3} & 0 & 0 \\ e^{i\theta_1} & 0 & e^{-i\theta_4} & 0 & e^{-i\theta_5} & e^{-i\theta_6} \\ e^{i\theta_2} & e^{i\theta_4} & 0 & e^{-i\theta_7} & 0 & 0 \\ e^{i\theta_3} & 0 & e^{i\theta_7} & 0 & e^{-i\theta_8} & e^{-i\theta_9} \\ 0 & e^{i\theta_5} & 0 & e^{i\theta_8} & 0 & 0 \\ 0 & e^{i\theta_6} & 0 & e^{i\theta_9} & 0 & 0 \end{pmatrix}$$

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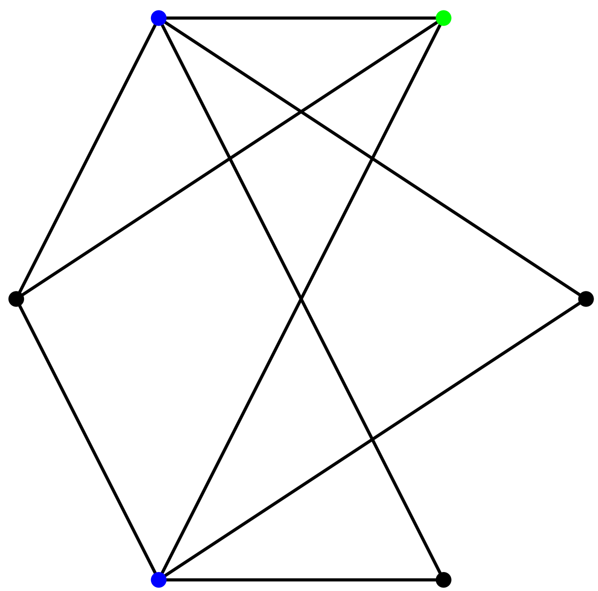

$U = e^{-iPt}$

Random phases give different dynamics.

$$t = 0.5$$

{0.438, 0.239, 0.107, 0.173, 0.0012, 0.040}

$$\begin{pmatrix} 0 & e^{-i\theta_1} & e^{-i\theta_2} & e^{-i\theta_3} & 0 & 0 \\ e^{i\theta_1} & 0 & e^{-i\theta_4} & 0 & e^{-i\theta_5} & e^{-i\theta_6} \\ e^{i\theta_2} & e^{i\theta_4} & 0 & e^{-i\theta_7} & 0 & 0 \\ e^{i\theta_3} & 0 & e^{i\theta_7} & 0 & e^{-i\theta_8} & e^{-i\theta_9} \\ 0 & e^{i\theta_5} & 0 & e^{i\theta_8} & 0 & 0 \\ 0 & e^{i\theta_6} & 0 & e^{i\theta_9} & 0 & 0 \end{pmatrix}$$

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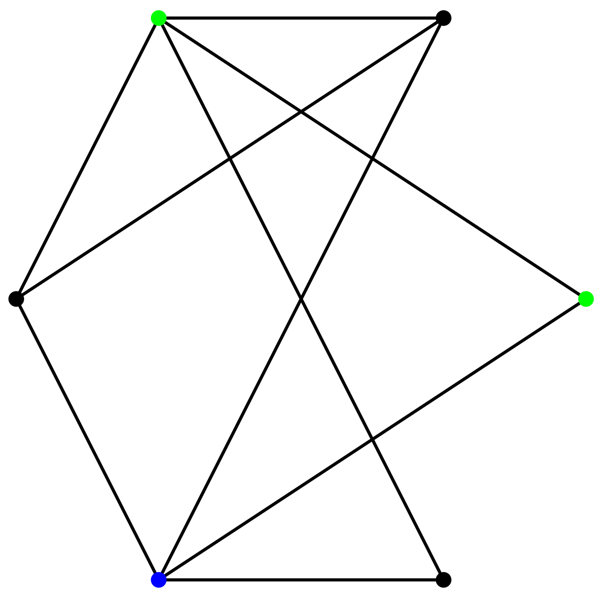

$U = e^{-iPt}$

Random phases give different dynamics.

$$t = 1.0$$

{0.005, 0.367, 0.028, 0.216, 0.047, 0.335}

$$\begin{pmatrix} 0 & e^{-i\theta_1} & e^{-i\theta_2} & e^{-i\theta_3} & 0 & 0 \\ e^{i\theta_1} & 0 & e^{-i\theta_4} & 0 & e^{-i\theta_5} & e^{-i\theta_6} \\ e^{i\theta_2} & e^{i\theta_4} & 0 & e^{-i\theta_7} & 0 & 0 \\ e^{i\theta_3} & 0 & e^{i\theta_7} & 0 & e^{-i\theta_8} & e^{-i\theta_9} \\ 0 & e^{i\theta_5} & 0 & e^{i\theta_8} & 0 & 0 \\ 0 & e^{i\theta_6} & 0 & e^{i\theta_9} & 0 & 0 \end{pmatrix}$$

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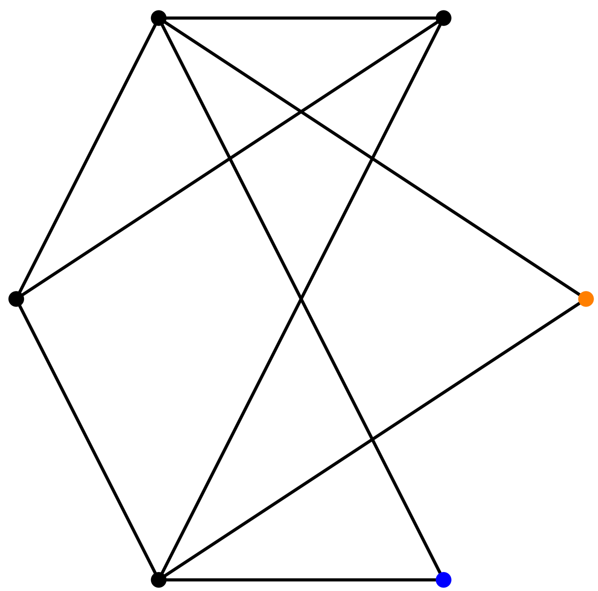


$U = e^{-iPt}$

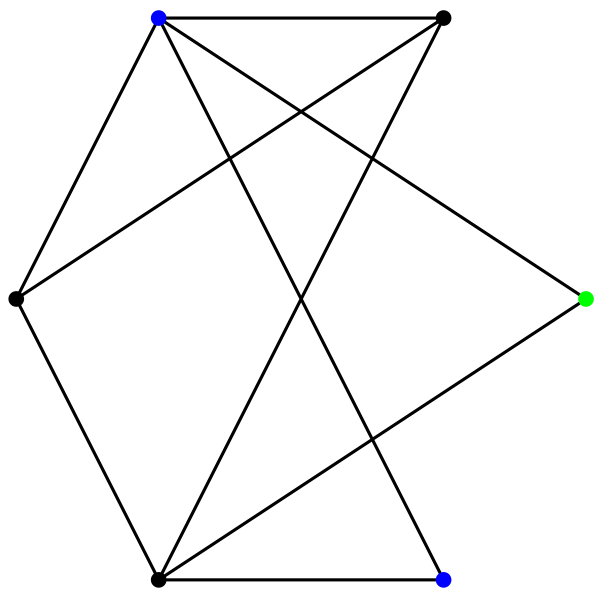
Random phases give different dynamics.

$$t = 1.5$$

{0.076, 0.047, 0.077, 0.032, 0.211, 0.554}

$$\begin{pmatrix} 0 & e^{-i\theta_1} & e^{-i\theta_2} & e^{-i\theta_3} & 0 & 0 \\ e^{i\theta_1} & 0 & e^{-i\theta_4} & 0 & e^{-i\theta_5} & e^{-i\theta_6} \\ e^{i\theta_2} & e^{i\theta_4} & 0 & e^{-i\theta_7} & 0 & 0 \\ e^{i\theta_3} & 0 & e^{i\theta_7} & 0 & e^{-i\theta_8} & e^{-i\theta_9} \\ 0 & e^{i\theta_5} & 0 & e^{i\theta_8} & 0 & 0 \\ 0 & e^{i\theta_6} & 0 & e^{i\theta_9} & 0 & 0 \end{pmatrix}$$

Justifying Phased Walks



$U = e^{-iPt}$

Random phases give different dynamics.

$$t = 2.0$$

{0.025, 0.190, 0.098, 0.093, 0.268, 0.324}

$$\begin{pmatrix} 0 & e^{-i\theta_1} & e^{-i\theta_2} & e^{-i\theta_3} & 0 & 0 \\ e^{i\theta_1} & 0 & e^{-i\theta_4} & 0 & e^{-i\theta_5} & e^{-i\theta_6} \\ e^{i\theta_2} & e^{i\theta_4} & 0 & e^{-i\theta_7} & 0 & 0 \\ e^{i\theta_3} & 0 & e^{i\theta_7} & 0 & e^{-i\theta_8} & e^{-i\theta_9} \\ 0 & e^{i\theta_5} & 0 & e^{i\theta_8} & 0 & 0 \\ 0 & e^{i\theta_6} & 0 & e^{i\theta_9} & 0 & 0 \end{pmatrix}$$

Justifying Phased Walks

What are the upsides of phased walks?

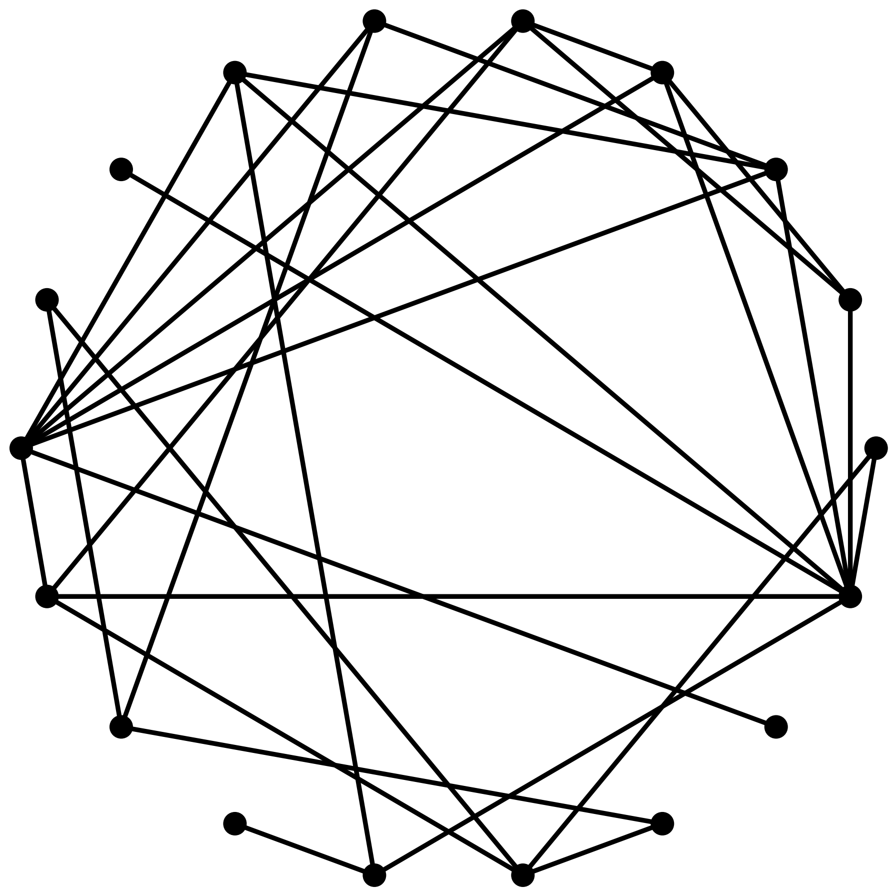
- Does not need an extra coin space. For N vertices we only need N states
- May allow new interesting behaviors
  - Speed up spatial search?
  - Different limiting distributions

What are the downsides?

- More complicated to analyze.
- May not allow new interesting behaviors.
- Why hasn't anyone else done it?

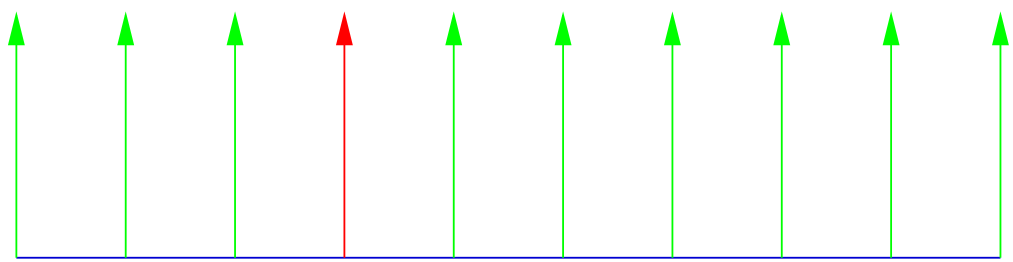
Spatial Search

Our data is arranged in space according to some graph, it takes unit time to move between the vertices.



Spatial Search

Grover search works on unstructured databases. Can it also work on spatially arranged databases?



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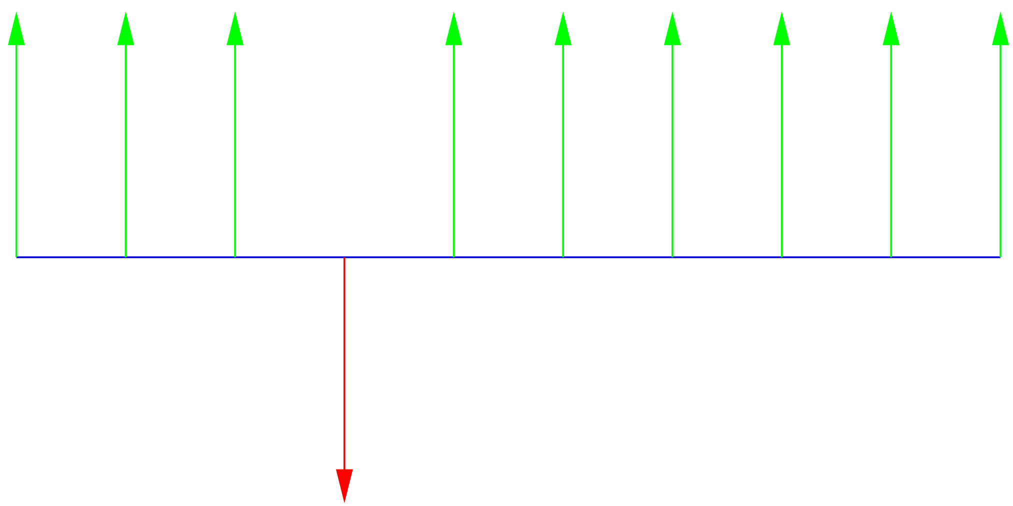

Phase flip

Spatial Search

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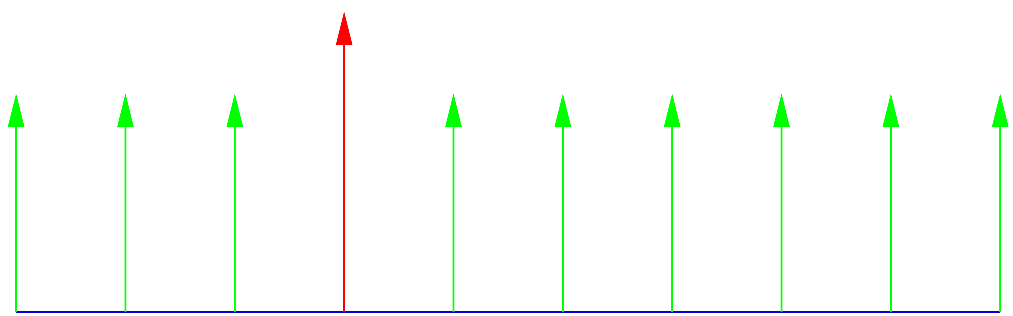

Reflect about mean

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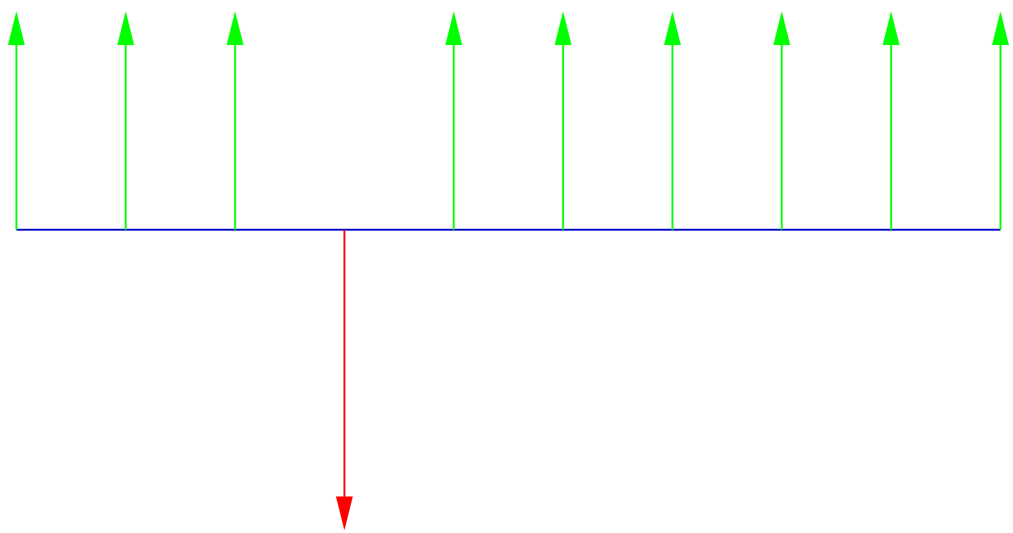

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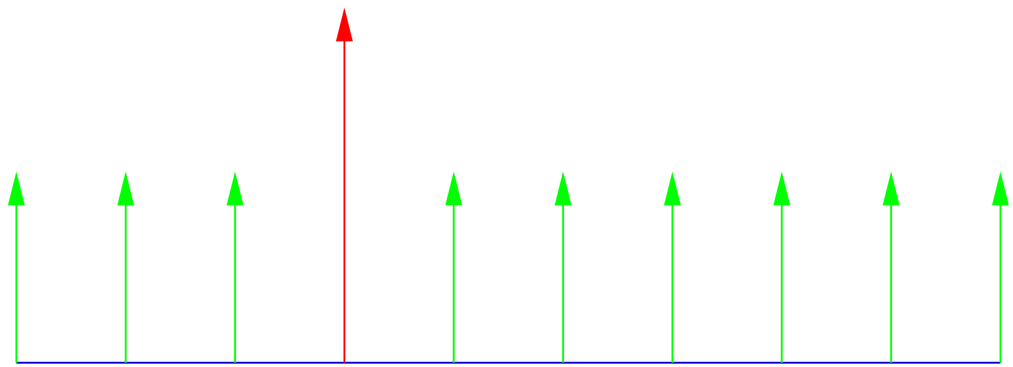
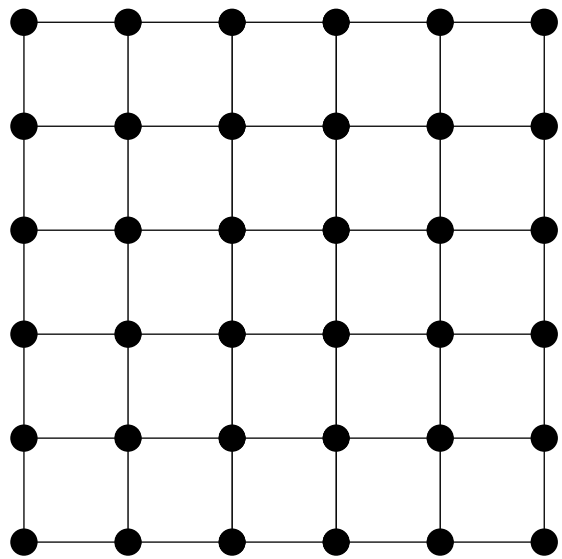


Reflect about mean

Benioff's Claim

The 2D grid has diameter $O(\sqrt{N})$, so grover search takes time $O(\sqrt{N} * \sqrt{N}) = O(N)$.



In 2001 Benioff suggested that for the two dimensional grid quantum search is no better than classical.

Maybe sorting is $\Omega(N^2)$ too??

Benioff is Not a Computer Scientist

Divide and Conquer to the Rescue!

- Aaronson and Ambainis (2003) show that the 2D grid can be searched in $O(\sqrt{N} log^3 N)$ using recursion and amplitude amplification.
- They also show that for $d > 2$ the d dimensional grid can be searched in $O(\sqrt{N})$. Among other things...

Lower Bounds

It is not obvious that $\Omega(\sqrt{N})$ is the lower bound on spatial search on the d dimension grid. The best we have:

d	lower bound
2	$\Omega(N^{1/6})$
3	$\Omega(N^{1/4})$
4	$\Omega(N^{3/10})$
5	$\Omega((N/logN)^{1/3})$
> 6	$\Omega(N^{1/3})$

[Zhang 2005]

Related Results

- Farhi and Gutmann (1996) Showed that the complete graph can be searched in $O(\sqrt{N})$ by a continuous-time quantum walk. This is equivalent to grover search.
- Shenvi, Kempe, Whaley (2002) Showed that the hypercube can be searched by a discrete time quantum walk in $O(\sqrt{N})$.
- Childs and Goldstone (2003) Showed the following for searching the d dimensional grid.

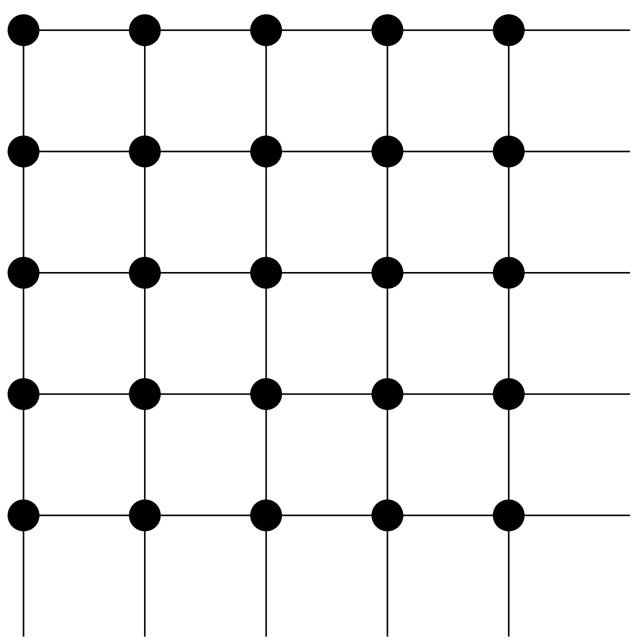
$d < 4$	$O(N)$
$d = 4$	$O(\sqrt{N} log N)$
$d > 4$	$O(\sqrt{N})$

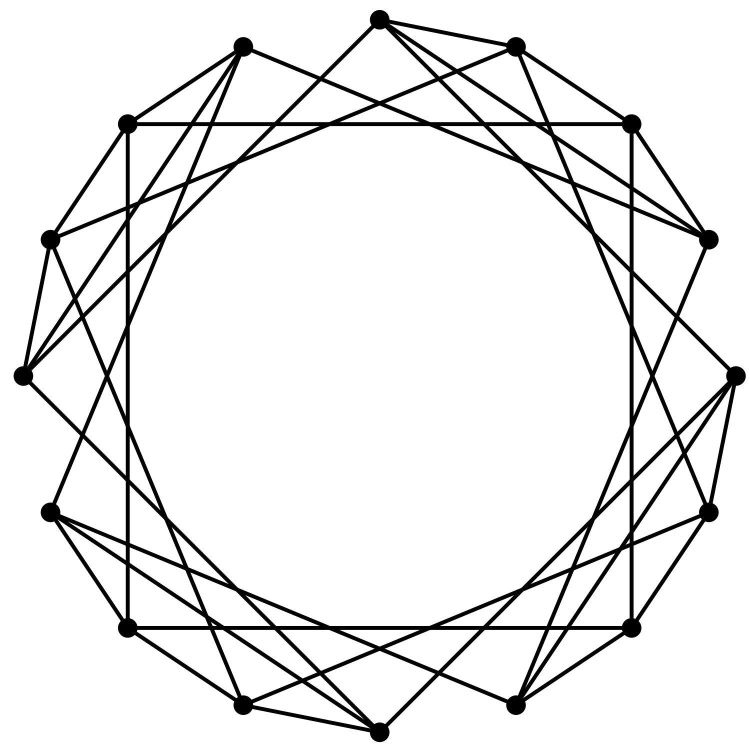
Related Results

- Ambainis, Kempe, Rivosh (2004) Showed the grid can be searched by a dtqw in $O(\sqrt{N})$ for $d > 2$ and $O(\sqrt{N} logN)$ for $d = 2$. This was the first (only?) result showing a separation between dtqw's and ctqw's.
- Childs and Goldstone (2004) modify the ctqw model by using a Hamiltonian with spin states and used it to search the grid in $O(\sqrt{N})$ for $d > 2$ and $O(\sqrt{N} logN)$ for $d = 2$. $2^{d/2}$ spin states are needed, but this is ok for low dimensions $\{2, 4, 4\}$ states for $d = \{2, 3, 4\}$.
- Ambainis's algorithm (2004) for element distinctness uses a dtqw to search special graphs.

The Grid

When we are doing quantum walks on the grid we are actually walking on cubic periodic lattices which are periodic in every direction... Or in 2D we are walking on the discrete version of the torus:





Pros and Cons of the Models

CTQW's

- Pro:Simple
- Con: Slow in low dimensions
- Con: Unknown performance on many graphs

DTQW's

- Pro: Fast
- Con: Extra states, extra complexity
- Con: Unknown performance on many graphs

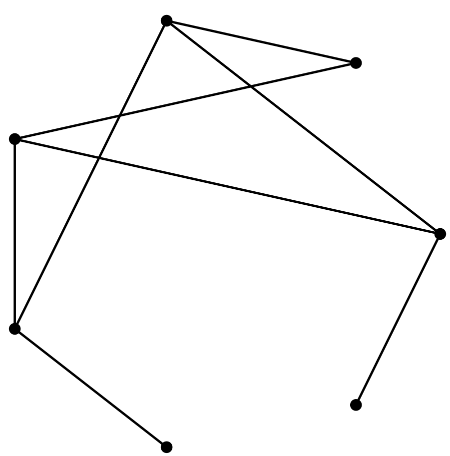
Amplitude Amplification

- Pro: Fast
- Pro: Works well on a wide rage of graphs
- Con: Complex

Spatial Search by CTQW

Childs and Goldstones idea:

- Start walk in $|s\rangle = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}|j\rangle$
- Run Hamiltonian $H = \gamma(D - A) - |w\rangle\langle w|$, where $|w\rangle$ is the marked state and γ is an adjustable parameter.



$$\begin{pmatrix} 2\gamma & -\gamma & -\gamma & 0 & 0 & 0 & 0 \\ -\gamma & 3\gamma & 0 & -\gamma & 0 & 0 & -\gamma \\ -\gamma & 0 & 3\gamma - 1 & -\gamma & 0 & 0 & -\gamma \\ 0 & -\gamma & -\gamma & 3\gamma & -\gamma & 0 & 0 \\ 0 & 0 & 0 & -\gamma & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma & -\gamma \\ 0 & -\gamma & -\gamma & 0 & 0 & -\gamma & 3\gamma \end{pmatrix}$$

Guiding Principle

Why does it work?

- $|s\rangle$ is the ground state of $D - A$, $|w\rangle$ is the ground state of $-|w\rangle\langle w|$.
- as γ goes from 0 to 1 the ground state of $H = \gamma(D - A) - |w\rangle\langle w|$ will switch from $|w\rangle$ to $|s\rangle$.
- If we assume that $|s\rangle$ and $|w\rangle$ both have substantial overlap with the first excited state of H then perturbation theory says that H will drive transitions between them in time of order $1/(E_1 - E_0)$.
- E_1 and E_0 are the first (lowest) two eigenvalues of H, $E_1 - E_0$ is the eigenvalue gap.

Spatial Search and Phased CTQW's

The previous argument is directly applicable to phased ctqw's. The only modification is that we will start the walk at $|\phi_0\rangle$ the ground state eigenvector of the phased adjacency matrix. The question is then:

Can we increase the eigenvalue gap?

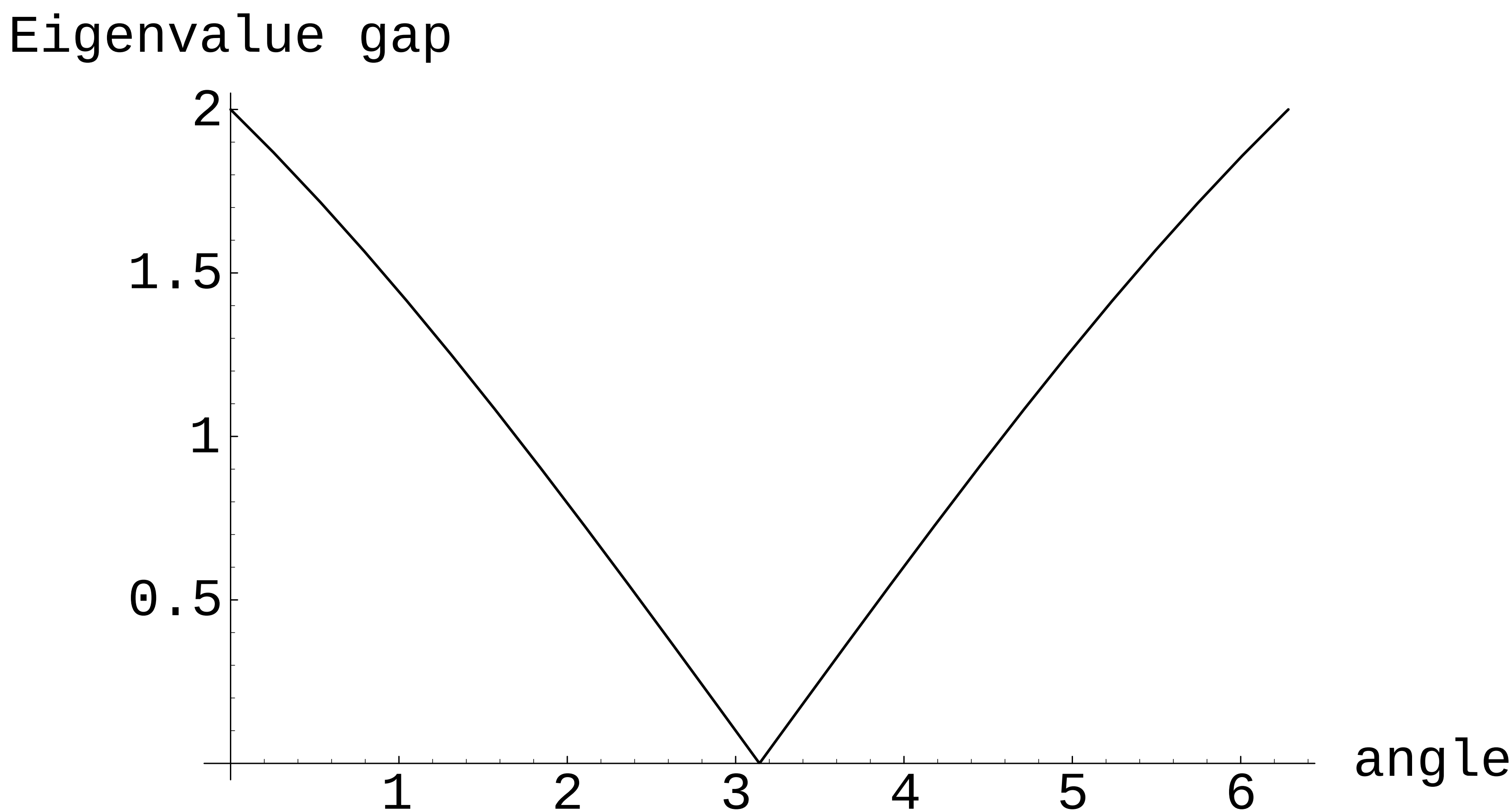
In general this seems to be a hard question as we are adding as many phase variable as there are edges. Also, randomly choosing the phases does not help on many graphs. Random graphs with random phases do not seem to work either.

The Grid

- For the grid we can get "close" to the full k (the number of edges) phases by noting that the grid (periodic version) is the graph Cartesian product of two cycles.
- This means the eigenvalues of the grid are simply sums of pairs of eigenvalues of the cycle. This can easily be proved to be true for the phased grid (or any phased graph).
- This allows us to introduce $O(\sqrt{k})$ phases and still analyze the walk.

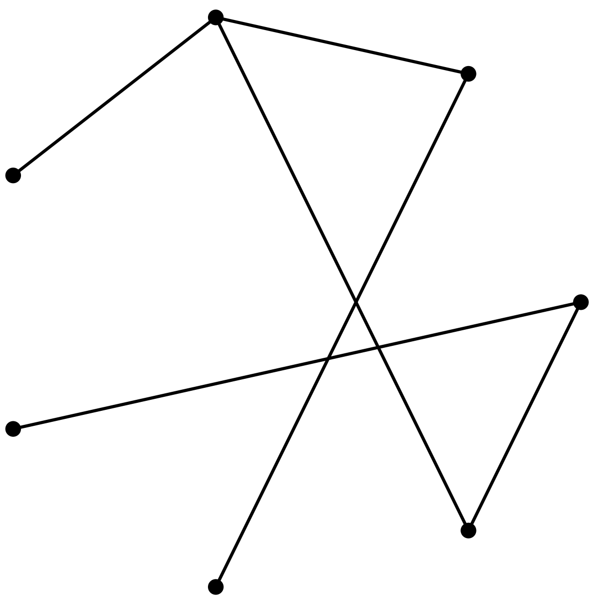
The Grid Sucks

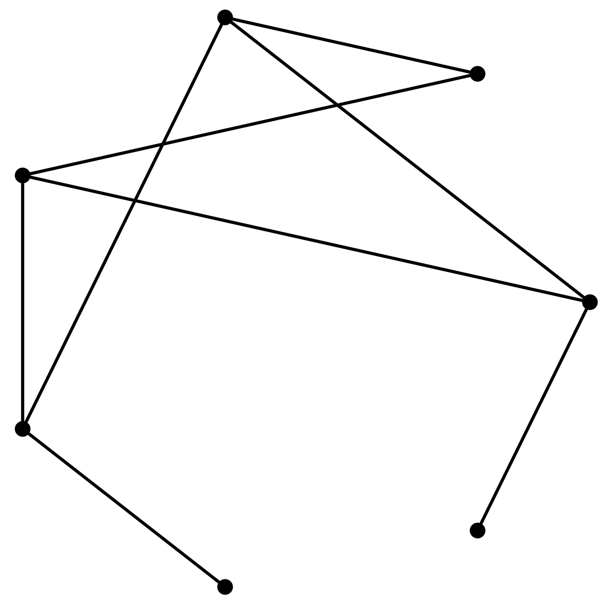
Unfortunately when we do this it seems like phases can only slow things down:



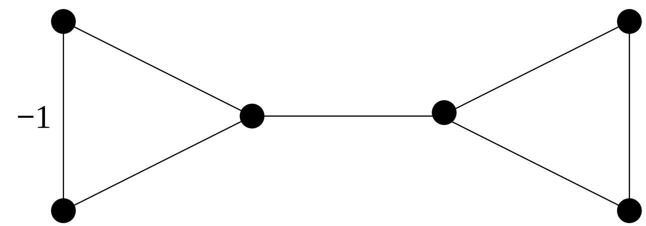
Cycles

Numerical investigations point to the fact that phases will cancel unless they are in a cycle. The graph on the left has the same eigenvalues as the unphased graph, while this is not true of the graph on the right.

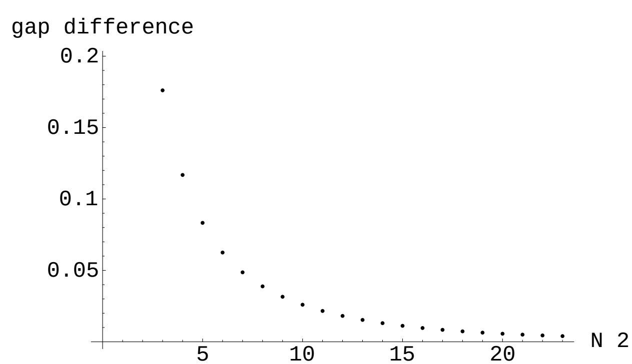




Linked Cycles



- The unphased eigenvalue gap of this graph is 0.682, with a phase of -1 at the position shown the gap is 1.236, an increase of 81%!
- That seems pretty cool, but how does the difference change as the size of the cycles grows?



Search Times

So the gap is bigger on small instances but is it really faster? The Times for the 3×3 double cycle:

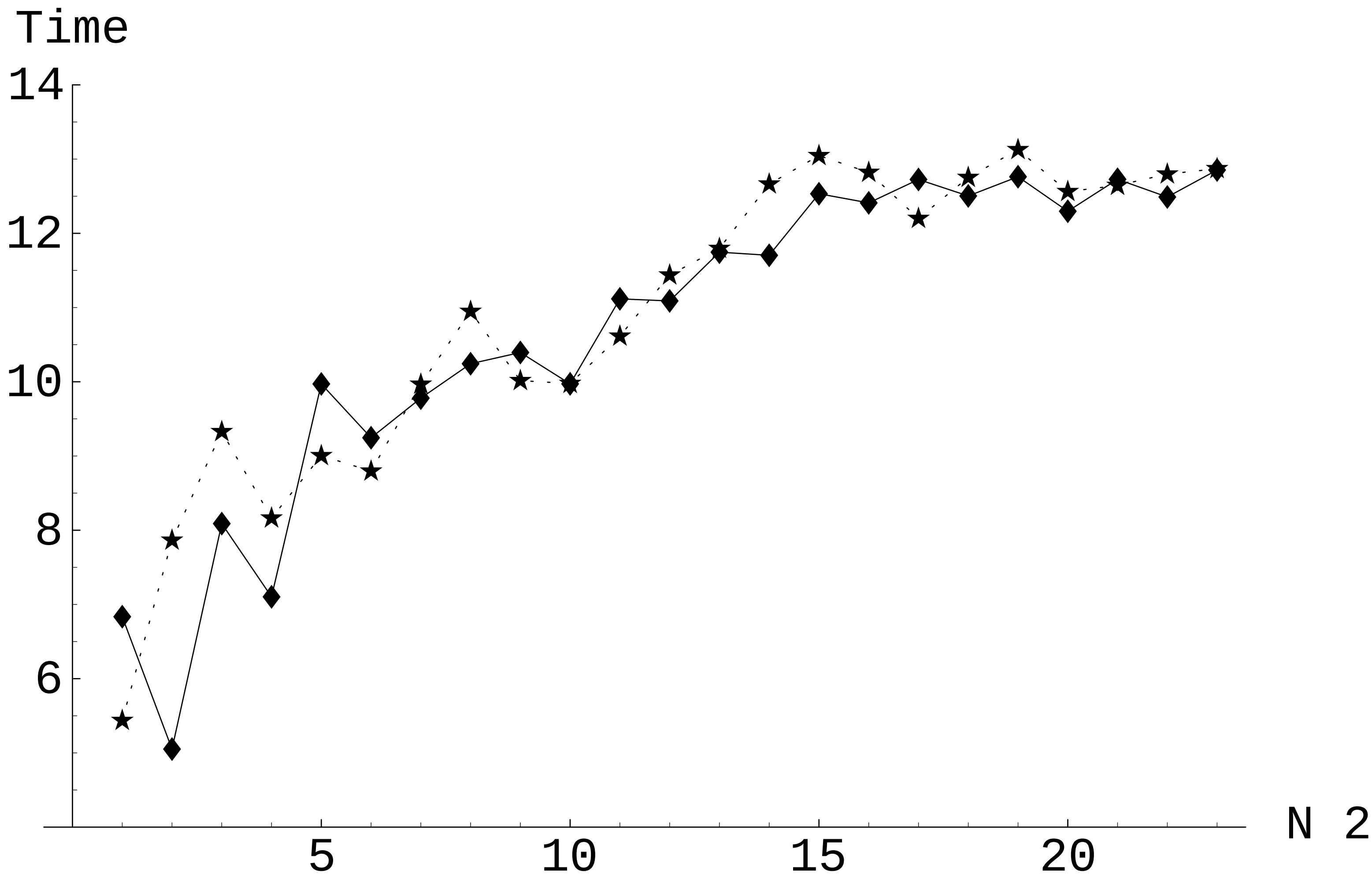
- Phased: {15.1, 15.1, 5.41, 1.54, 1.93, 1.93}
- Unphased: {6.33, 6.33, 6.06, 6.06, 6.33, 6.33}
- Averages p/u: 6.84/6.24

It is obvious that the eigenvalues are not the only things we have to worry about anymore. The eigenvectors may now be important.

$$|\phi_0\rangle = \{0.115, -0.115, -0.372, -0.602, -0.487, -0.487\}$$

Average Search Times

How long does it take to search on average?



Things To Think About

- Are there graphs with increasing eigenvalue gap difference?
- Do the eigenvectors really matter as the graphs get larger?
- My example is very contrived, what about higher degree graphs like the grid or random graphs?
- Can regular graphs be sped up?
- How about applying phased ctqw's to other things?
- Why hasn't anyone else done this?